Problem set 01

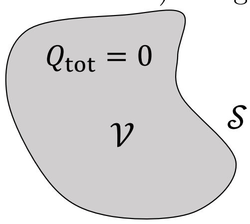
Nonlinear Optics for Quantum Technologies

February 20, 2025 (C. Galland, V. Goblot; EPFL)

Volumic polarisation in a medium (electrostatic)

Reminder: in electrostatic, $\vec{\nabla} \times \vec{E} = \vec{0}$ and $\vec{\nabla} \cdot (\varepsilon_0 \vec{E}) = \rho(\vec{r})$, with $\vec{E}$ the electric field, ρ the electric charge density and ε_0 the vacuum permittivity.

We consider a globally neutral medium of volume $\mathcal{V}$ and surface $\mathcal{S}$. We note ρ_{in} the charge density inside the medium. ρ_{in} refers here to "internal" charges, which are by definition the charges naturally present inside the medium, which simply react to the applied field. These can be either bound (in dielectrics) or free (in metals and doped semiconductors) charges.



1. Write the equation expressing the global neutrality of the medium.

We now define the dipole moment of this charge distribution (as seen from a large distance to this medium): $\vec{\mathcal{P}} = \int_{\mathcal{V}} \vec{r} \rho_{in}(\vec{r}) \mathrm{d}\mathcal{V}$

2. Prove that

$$\vec{\mathcal{P}} = \int_{\mathcal{V}} \vec{P}(\vec{r}) d\mathcal{V} \tag{1}$$

where $\vec{P}(\vec{r})$ is a field that satisfies:

$$\vec{\nabla} \cdot \vec{P}(\vec{r}) = -\rho_{in}(\vec{r}) \quad \text{if} \quad \vec{r} \in \mathcal{V} \tag{2}$$

$$\vec{P}(\vec{r}) = \vec{0} \quad \text{if} \quad \vec{r} \notin \mathcal{V} \tag{3}$$

$\vec{P}(\vec{r})$ is called a volumic polarisation density

Hints:

- Prove the relation for each x, y, z component independently (e.g., $\mathcal{P}_x = \int_{\mathcal{V}} \vec{P}_x(\vec{r}) d\mathcal{V}$)
- Use the formula:

$$\vec{\nabla} \cdot (f\vec{A}) = f \vec{\nabla} \cdot \vec{A} + \vec{\nabla} f \cdot \vec{A} \tag{4}$$

- Reminder: $\int_{\mathcal{V}'} \vec{\nabla} \cdot \vec{A} d\mathcal{V}' = \oint_{\mathcal{S}'} \vec{A} \cdot \vec{n} d\mathcal{S}'$ where $\mathcal{S}'$ is the surface enclosing $\mathcal{V}'$

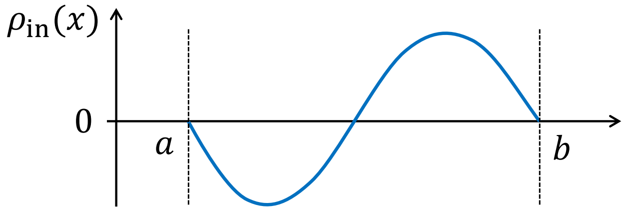
3. Are the following relations correct?

$$\vec{\nabla} \cdot \vec{P} = -\epsilon_0 \vec{\nabla} \cdot \vec{E} \tag{5}$$

$$\vec{P} = -\epsilon_0 \vec{E} \tag{6}$$

4. What is the SI unit of $\vec{P}(\vec{r})$?

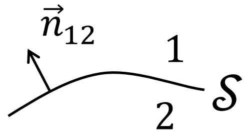
5. The figure below describes schematically a one-dimensional charge distribution $\rho_{in}(x)$ inside a globally neutral medium spanning from a to b:



Plot the shape of $P(x)$.

6. We now consider that external charges may exist, with a corresponding charge density $\rho_{ex}(\vec{r})$, so that $\rho(\vec{r}) = \rho_{in}(\vec{r}) + \rho_{ex}(\vec{r})$. These charges are those that can be controlled by an independent mean such as a gate, a wire, etc. Define the electric flux density $\vec{D}(\vec{r})$, satisfying $\vec{\nabla} \cdot \vec{D} = \rho_{ex}(\vec{r})$

7. We consider the interface between medium 1 and 2:



Using Gauss theorem, prove the two interface conditions :

$$\vec{n}_{12} \times \left(\vec{E}_2 - \vec{E}_1\right) = \vec{0} \tag{7}$$

$$\vec{n}_{12} \cdot \left(\vec{D}_2 - \vec{D}_1\right) = 0 \tag{8}$$

8. How is Eq. (8) modified in case of an extra surface charge density σ_{ext} ?

9. The electrostatic potential created by the charge distribution $\rho_{in}(\vec{r})$ is (considering $\rho_{ex} = 0$):

$$V_{el}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho_{in}(\vec{r'})}{||\vec{r} - \vec{r'}||} d\mathcal{V} \tag{9}$$

Using the same mathematical identities as in question 2, express $V_{el}(\vec{r})$ as an integral involving $\vec{P}(\vec{r'})$ over $\mathcal{V}$. What can you say about this expression ?

10. Show that, far from the object (i.e, $r \gg r'$ for the origin inside the medium):

$$V_{el}(\vec{r}) \simeq \frac{1}{4\pi\epsilon_0} \frac{\vec{\mathcal{P}} \cdot \vec{r}}{r^3} \tag{10}$$

where we recall that $\vec{\mathcal{P}} = \int_{\mathcal{V}} \vec{P}(\vec{r}) \mathrm{d}\mathcal{V}$.

Home exercise: dispersion relation

1. Recall Maxwell's equations in vacuum and give the name of the four corresponding laws in electrostatic.
2. Using the vector identity $\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E}$ derive the wave equation for the electric field.
3. Establish the dispersion relation of electromagnetic plane waves in vacuum (angular frequency ω as a function of wave-vector k).
4. Do the same in an isotropic medium whose induced polarisation density reacts instantaneously to the background field, i.e., $\boldsymbol{P}(t) = \varepsilon_0 \chi^{(1)} \boldsymbol{E}(t)$. How is the dispersion relation modified? How does the phase velocity depend on frequency? Is it a good description of a transparent material such as glass?
5. How would you modify the above relation to model a medium with chromatic dispersion?